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Crack branching in thermopiezoelectric materials

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ABSTRACT

Solutions are presented for an electrically impermeable crack branching out of the crack plane in a thermopiezoelectric medium under thermo-electro-mechanical loads based on Stroh formalism. Explicit Green's functions for the interaction of a crack and a thermopiezoelectric dislocation (i.e., a thermal dislocation, a mechanical dislocation and an electric dipole located at the same point) are developed. The problem then can be expressed in terms of coupled singular integral equations for the thermopiezoelectric dislocation density functions associated with a branched crack. Some essential fracture mechanics parameters, such as stress and electric displacement intensity factors, and energy release rate at the branched crack tip are obtained. Numerical results are presented for the effect of applied thermal flux loads and electric field on the crack propagation path.

1. Introduction

Piezoelectric materials have been widely used in engineering structures because the intrinsic coupling between the mechanical and the electric fields. They can fail prematurely due to defects such as cracks arising in the manufacturing process when subjected to mechanical, electric and thermal load. An analysis of the fracture process of these materials could provide useful information to improve the design of piezoelectric devices. There are numerous contributions in the literature on the fracture mechanics of piezoelectric materials, see Parton (1976), Pak (1990), Sosa (1992), Suo et al. (1992), Qin (1998a), Zhang et al. (1998), Wang and Mai (2004), Loboda et al. (2007) and Ueda et al. (2012).

On the other hand, crack branching plays an important role in the fracture of piezoelectric materials in response to the thermo-electro-mechanical load. The direction of crack branching can be one of the major factors in determining the residual strength of the structural components. Incipient crack branching in piezoelectric ceramics subjected to an electric field was first reported by McHenry and Koepke (1983). Furuta and Uchino (1993) found that branched crack propagation in multilayer piezoelectric actuators made of $Pb((Ni_{1/3}Nb_{2/3}),Ti,Zr)O_3$. Park and Sun (1995a) reported that the crack propagation deviated from its original direction under the combined mechanical and electrical load in their three-point bending test with an unsymmetrical crack in a PZT-4 specimen. The problem of crack kinking in a piezoelectric solid was investigated by Zhu and Yang (1999) by continuous distribution of edge dislocations and electric dipoles method. A solution for a class of two-dimensional electroelastic branched crack problems in various bimaterial combinations was presented by Qin and Mai (2000). The effect of a transverse electric field on crack kinking in ferroelectric ceramics subjected to purely electrical load was investigated by Jeong et al. (2008). It is note that another very important case of crack deflection in bimaterial systems with various materials combinations was solved by Qin and Zhang (2000).

However, to the best knowledge of the authors so far, study of the crack branching or kinking in thermopiezoelectric materials is very limited. In the current paper, the extended Stroh's formalism of anisotropic thermopiezoelectric medium combined with singular integral equation approach are used to determine the stress and electric displacement (SED) intensity factors and energy release rate for arbitrary branch angle. The plan of the paper is as follows. In Section 2 we outline the basic theory of the Stroh formalism. In Section 3 a closed form solution is obtained for the interaction between a crack and a thermopiezoelectric dislocation. Sections 4 and 5, model the crack branched portion by a continuous distribution of thermopiezoelectric dislocations, leading to two sets of coupled singular integral equations in terms of thermal dislocation density and piezoelectric dislocation density. Some numerical results are presented in Section 6, and concluding remarks are made in Section 7.

2. The Stroh formalism

Consider a linear piezoelectric material in which all fields are assumed to depend only on the in-plane coordinates x_1 and x_2. The shorthand notation developed by Barnett and Lothe (1975)

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based upon Stroh's formalism (Stroh, 1958) is adopted in this paper. In the stationary case when no free electric charge, body force or heat source exists, the basic equations for thermopiezoelectric materials can be written as (Mindlin, 1974)

$$h_{i,i} = 0, \quad \mathbf{\Pi}_{iJ,i} = 0 \tag{1}$$

together with

$$h_i = -k_{ij}T_{,j}, \quad \mathbf{\Pi}_{iJ} = E_{iJKm}u_{k,m} - \chi_{iJ}T \tag{2}$$

in which

$$\mathbf{\Pi}_{iJ} = \begin{cases} \sigma_{ij}, & i,J = 1,2,3 \\ D_i, & i = 1,2,3;\ J = 4 \end{cases} \tag{3}$$

$$u_J = \begin{cases} u_k, & k = 1,2,3 \\ \phi, & J = 4 \end{cases} \tag{4}$$

$$\chi_{iJ} = \begin{cases} \beta_{ij}, & i,J = 1,2,3 \\ \gamma_i & i = 1,2,3;\ J = 4 \end{cases} \tag{5}$$

$$E_{iJKm} = \begin{cases} c_{ijkm}, & i,J,K,m = 1,2,3 \\ e_{mij}, & i,J,m = 1,2,3;\ K = 4 \\ e_{ikm}, & i,K,m = 1,2,3; J = 4 \\ -\kappa_{im}, & J = K = 4;\ i,m = 1,2,3 \end{cases} \tag{6}$$

where T and h_i are temperature change and heat flux, u_i, ϕ, σ_{ij} and D_i are elastic displacement, electric potential, stress and electric displacement, c_{ijkm}, e_{ijk} and κ_{ij} are elastic moduli, piezoelectric and dielectric constants, k_{ij}, β_{ij} and γ_i are coefficients of heat conduction, thermal-stress constants and pyroelectric constants, respectively. The general solution to the Eq. (1) can be written as (Qin, 1998b)

$$\begin{aligned} T &= g'(z_t) + \overline{g'(z_t)} \\ \mathbf{u} &= \mathbf{Af}(z)\mathbf{q} + \mathbf{c}g(z_t) + \overline{\mathbf{Af}(z)\mathbf{q}} + \overline{\mathbf{c}g(z_t)} \end{aligned} \tag{7}$$

with $\mathbf{A} = [A_1,\ A_2,\ A_3,\ A_4]$, $\mathbf{f}(z) = diag[f(z_1),\ f(z_2),\ f(z_3),\ f(z_4)]$, $\mathbf{q} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}}$, $z_t = x_1 + p_* x_2$, $z_i = x_1 + p_i x_2$, in which the prime denotes differentiation with the argument, the overbars denote complex conjugation, $\mathbf{q}$ is a constant vector to be determined by the boundary conditions, g and $\mathbf{f}$ are arbitrary analytic function, p_*, p_i, $\mathbf{A}$ and $\mathbf{c}$ are constants determined by

$$\begin{aligned} &k_{11} + 2k_{12}p_* + k_{22}p_*^2 = 0 \\ &[\mathbf{Q} + p_i(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p_i^2\mathbf{T}]\mathrm{A}_i = 0 \\ &[\mathbf{Q} + p_*(\mathbf{R} + \mathbf{R}^{\mathrm{T}}) + p_*^2\mathbf{T}]\mathbf{c} = \boldsymbol{\chi}_1 + p_*\boldsymbol{\chi}_2 \end{aligned} \tag{8}$$

in which superscript "$\mathbf{T}$" denotes the transpose, $\boldsymbol{\chi}_i$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are defined by

$$\boldsymbol{\chi}_i = [\beta_{i1}, \beta_{i2}, \beta_{i3}, \gamma_i,]^{\mathrm{T}}, \quad \mathbf{Q}_{IK} = E_{1IK1}, \quad \mathbf{R}_{IK} = E_{1IK2}, \quad \mathbf{T}_{IK} = E_{2IK2} \tag{9}$$

The heat flux, h, and stress and electric displacement, $\mathbf{\Pi}_{iJ}$, can be obtained from Eq. (2) as

$$\begin{aligned} h_i &= -(k_{i1} + p_*k_{i2})g''(z_t) - (k_{i1} + \overline{p_*}k_{i2})\overline{g''(z_t)} \\ \mathbf{\Pi}_{1J} &= -\mathbf{\Phi}_{J,2}, \quad \mathbf{\Pi}_{2J} = \mathbf{\Phi}_{J,1} \end{aligned} \tag{10}$$

where $\mathbf{\Phi}$ is the SED function give as

$$\mathbf{\Phi} = \mathbf{Bf}(z)\mathbf{q} + \mathbf{d}g(z_t) + \overline{\mathbf{Bf}(z)\mathbf{q}} + \overline{\mathbf{d}g(z_t)} \tag{11}$$

with

$$\begin{aligned} \mathbf{B} &= \mathbf{R}^{\mathrm{T}}\mathbf{A} + \mathbf{TAP} \\ \mathbf{P} &= diag[p_1,\ p_2,\ p_3,\ p_4] \\ \mathbf{d} &= (\mathbf{R}^{\mathrm{T}} + p_*\mathbf{T})\mathbf{c} - \boldsymbol{\chi}_2 \end{aligned} \tag{12}$$

Let $k = k_{22}(p_* - \overline{p_*})/2i$, then $k = \sqrt{k_{11}k_{22} - k_{12}^2}$ and

$$\begin{aligned} h_1 &= ikp_*g''(z_t) - ik\overline{p_*g''(z_t)} \\ h_2 &= -ikg''(z_t) + ik\overline{g''(z_t)} \end{aligned} \tag{13}$$

3. Crack–dislocation interaction in anisotropic thermopiezoelectric media

3.1. The thermoelectroelastic Green's function

The temperature discontinuity can be modeled by the thermal analog of a line dislocation, and the Green's function for the thermo-elastic field (heat vortex) in an anisotropic media was addressed by Sturia and Barber (1998). Here, a new thermoelectroelastic Green's function will be derived based on their work. A cut is assumed to be located at $z = z_0(x_{10}, x_{20})$ along the plane $(x_1 < x_{10},\ x_2 = x_{20})$ and a constant temperature discontinuity T_0 exists along this cut, i.e.,

$$T(x_1, x_2 = x_{20}^+) - T(x_1, x_2 = x_{20}^-) = T_0, \quad x_1 - x_{10} < 0 \tag{14}$$

The solution of Eq. (14) is of the form

$$T(x_1, x_2) = \frac{T_0}{4\pi i}[\ln(z_t - z_{0t}) - \ln(\bar{z}_t - \bar{z}_{0t})], \quad z_{0t} = x_{10} + p_*x_{20} \tag{15}$$

Therefore, using the Eqs. $(7)_1$, (10), $(13)_2$ and (15), the heat flux and SED fields induced by the temperature discontinuity can be written, respectively, as

$$h_2(x_1, x_2) = -\frac{kT_0}{4\pi}\left[\frac{1}{z_t - z_{0t}} + \frac{1}{\bar{z}_t - \bar{z}_{0t}}\right] \tag{16}$$

$$\mathbf{t}_2^t(x_1, x_2) = [\sigma_{21},\ \sigma_{22},\ \sigma_{23},\ D_2]^{\mathrm{T}} = \frac{T_0}{2\pi}\mathrm{Im}[\mathbf{d}\ \ln(z_t - z_{0t})] \tag{17}$$

In particular, on the $x_2 = 0$ plane we have

$$h_2(x_1, 0) = -\frac{kT_0}{2\pi}\mathrm{Re}\left[\frac{1}{x_1 - z_{0t}}\right] \tag{18}$$

$$\mathbf{t}_2^t(x_1, 0) = \frac{T_0}{2\pi}\mathrm{Im}[\mathbf{d}\ \ln(x_1 - z_{0t})] \tag{19}$$

3.2. Green's function of a piezoelectric dislocation

A piezoelectric crack can be modeled by a continuous distribution of mechanical dislocations and electric dipoles. Hereafter a mechanical dislocation and an electric dipole located at the same point are termed the piezoelectric dislocation. The solution of a single dislocation obtained by Suo (1990) in an anisotropic media can be expanded to piezoelectric material directly. Let the cut be the same as the one in the formation of the heat vortex, the displacement and stress function can be expressed in the form

$$\begin{aligned} \mathbf{u}^d &= \mathbf{A}\langle\ln(z_\alpha - z_{0\alpha})\rangle\mathbf{q}_0 + \overline{\mathbf{A}}\langle\ln(\bar{\mathbf{z}}_\alpha - \bar{\mathbf{z}}_{0\alpha})\rangle\overline{\mathbf{q}}_{\mathbf{0}} \\ \mathbf{\Phi}^d &= \mathbf{B}\langle\ln(z_\alpha - z_{0\alpha})\rangle\mathbf{q}_0 + \overline{\mathbf{B}}\langle\ln(\bar{\mathbf{z}}_\alpha - \bar{\mathbf{z}}_{0\alpha})\rangle\overline{\mathbf{q}}_{\mathbf{0}} \end{aligned} \tag{20}$$

where the superscript "d" denotes the value induced by the piezoelectric dislocation.

By definition of the piezoelectric dislocation one has

$$\mathbf{u}^d(\pi) - \mathbf{u}^d(-\pi) = \mathbf{b}, \quad \mathbf{\Phi}^d(\pi) - \mathbf{\Phi}^d(-\pi) = 0 \tag{21}$$

Solving for $\mathbf{q}$ from the above algebraic equations, one finds

$$\mathbf{q}_0 = \frac{1}{2\pi i}\mathbf{B}^{\mathrm{T}}\mathbf{b} \tag{22}$$

where the following normalized orthogonality relation

$$\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \overline{\mathbf{B}}^{\mathrm{T}} & \overline{\mathbf{A}}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \overline{\mathbf{A}} \\ \mathbf{B} & \overline{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{23}$$

is used.

Substituting Eqs. $(20)_2$ and (22) into Eq. $(10)_2$, the SED field induced by the edge piezoelectric dislocation alone at z_0 is of the form

$$\mathbf{t}_2^d(x_1,x_2)=\frac{1}{2\pi i}\mathbf{B}\left\langle\frac{1}{z_\alpha-z_{0\alpha}}\right\rangle\mathbf{B}^{\mathrm{T}}\mathbf{b}-\frac{1}{2\pi i}\overline{\mathbf{B}}\left\langle\frac{1}{\overline{\mathbf{z}}_\alpha-\overline{\mathbf{z}}_{0\alpha}}\right\rangle\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{b} \tag{24}$$

In particular, the SED along the x_1-axis is

$$\mathbf{t}_2^d(x_1,0)=\frac{1}{2\pi i}\mathbf{B}\left\langle\frac{1}{x_1-z_{0\alpha}}\right\rangle\mathbf{B}^{\mathrm{T}}\mathbf{b}-\frac{1}{2\pi i}\overline{\mathbf{B}}\left\langle\frac{1}{\mathbf{x}_1-\overline{\mathbf{z}}_{0\alpha}}\right\rangle\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{b} \tag{25}$$

3.3. A thermopiezoelectric dislocation interacting with a crack

The interaction between a thermopiezoelectric dislocation and a traction-charge free crack provides a fundamental solution for the crack branching in a piezoelectric medium. Consider a main crack of length $2a$ in a two-dimensional infinite homogeneous piezoelectric material, as shown in Fig. 1. The origin of the rectangular coordinate system x_1–x_2 is fixed at the crack center. The x_1-axis coincides with the crack and the poling direction of the material is along the x_2-axis. A heat flux $h(x_1)$ and $\mathbf{t}_0(x_1)$ is applied on the crack faces. The main crack is assumed to branch into $x_2>0$ (or $x_2<0$) at an angle θ with the x_1-axis. By the superposition principle, the boundary conditions for this problem can be expressed as

$$\begin{aligned}&h_2(x_1,x_2=0^+)=h_2(x_1,x_2=0^-)=-h(x_1),\quad -a\leqslant x_1\leqslant a, x_2=0\\&\mathbf{\Pi}_{2J}(x_1,x_2=0^+)=\mathbf{\Pi}_{2J}(x_1,x_2=0^-)=-\mathbf{t}_0(x_1),\quad -a\leqslant x_1\leqslant a,\ x_2=0\\&h_2=0,\quad \mathbf{\Pi}_{1J}=\mathbf{\Pi}_{2J}=0,\quad \text{at infinity}\end{aligned} \tag{26}$$

where the superscript " + " and "−" refer, respectively, to the upper and lower main crack surfaces.

Substitution of Eq. $(10)_1$ into Eq. $(26)_1$ yields

$$\begin{aligned}&-ikg''(x_1)^+ + ik\overline{g''}(x_1)^- = -h(x_1),\quad -a\leqslant x_1\leqslant a\\&-ikg''(x_1)^- + ik\overline{g''}(x_1)^+ = -h(x_1),\quad -a\leqslant x_1\leqslant a\end{aligned} \tag{27}$$

Leading to,

$$\begin{aligned}&[g''(x_1)+\overline{g''}(x_1)]^+-[g''(x_1)+\overline{g''}(x_1)]^-=0,\quad -a\leqslant x_1\leqslant a\\&[g''(x_1)-\overline{g''}(x_1)]^++[g''(x_1)-\overline{g''}(x_1)]^-=\frac{2h(x_1)}{ik},\quad -a\leqslant x_1\leqslant a\end{aligned} \tag{28}$$

Based on Muskhelishvili (1975) theorem and the assumption that $g''(z)$ vanishes at infinity, the solution of boundary value problem (28) can be obtained as

$$\begin{aligned}&g''(z)+\overline{g''}(z)=0\\&g''(z)-\overline{g''}(z)=\frac{X(z)}{2\pi i}\int_{-a}^{a}\frac{X(x_1)^+}{x_1-z}\frac{2h(x_1)}{ik}dx_1\end{aligned} \tag{29}$$

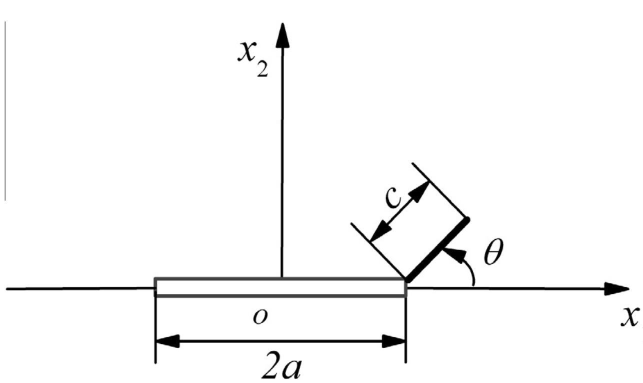

Fig. 1. A branched thermopiezoelectric crack.

where

$$X(z)=\frac{1}{\sqrt{z^2-a^2}} \tag{30}$$

Incorporating Eq. $(29)_{1,2}$ results in

$$g''(z)=\frac{X(z)}{2\pi i}\int_{-a}^{a}\frac{X(x_1)^+}{x_1-z}\frac{(-i)h(x_1)}{k}dx_1 \tag{31}$$

With the substitution of Eq. (18) into (31), the interaction function for SED and temperature distribution fields can be obtained as

$$g''_{\mathrm{int}}(z)=\frac{T_0}{4\pi i}\left[\frac{1}{\sqrt{z^2-a^2}}+\frac{\sqrt{z_{0t}^2-a^2}-\sqrt{z^2-a^2}}{2\sqrt{z^2-a^2}(z-z_{0t})}+\frac{\sqrt{\overline{z}_{0t}^2-a^2}-\sqrt{z^2-a^2}}{2\sqrt{z^2-a^2}(z-\overline{z}_{0t})}\right] \tag{32}$$

Integration of the above equation gives

$$g'_{\mathrm{int}}(z)=\frac{T_0}{4\pi i}B(z,z_{0t}) \tag{33}$$

with

$$\begin{aligned}B(z,z_{0t})=&\ln\left[z+\sqrt{z^2-a^2}\right]\\&-\frac{1}{2}\left\{\ln\left[\sqrt{z^2-a^2}+\frac{z_{0t}z-a^2}{\sqrt{z_{0t}^2-a^2}}\right]+\ln\left[\sqrt{z^2-a^2}+\frac{\overline{z}_{0t}z-a^2)}{\sqrt{\overline{z}_{0t}^2-a^2}}\right]\right\}\end{aligned} \tag{34}$$

where a constant which plays no role in the sequel section was omitted.

Next, from the boundary conditions $(26)_2$, we obtain

$$\begin{aligned}&\mathbf{B}\mathbf{f}'(x_1)^+ + \overline{\mathbf{B}\mathbf{f}'}(x_1)^- + \mathbf{d}g'(x_1)^+ + \overline{\mathbf{d}g'}(x_1)^- = -\mathbf{t}_0(x_1),\quad -a\leqslant x_1\leqslant a\\&\mathbf{B}\mathbf{f}'(x_1)^- + \overline{\mathbf{B}\mathbf{f}'}(x_1)^+ + \mathbf{d}g'(x_1)^- + \overline{\mathbf{d}g'}(x_1)^+ = -\mathbf{t}_0(x_1),\quad -a\leqslant x_1\leqslant a\end{aligned} \tag{35}$$

which leads to

$$\begin{aligned}&\left[\mathbf{B}\mathbf{f}'(x_1)-\overline{\mathbf{B}\mathbf{f}'}(x_1)+\mathbf{d}g'(x_1)-\overline{\mathbf{d}g'}(x_1)\right]^+\\&\quad-\left[\mathbf{B}\mathbf{f}'(x_1)-\overline{\mathbf{B}\mathbf{f}'}(x_1)+\mathbf{d}g'(x_1)-\overline{\mathbf{d}g'}(x_1)\right]^-=0\\&\left[\mathbf{B}\mathbf{f}'(x_1)+\overline{\mathbf{B}\mathbf{f}'}(x_1)+\mathbf{d}g'(x_1)+\overline{\mathbf{d}g'}(x_1)\right]^+\\&\quad+\left[\mathbf{B}\mathbf{f}'(x_1)+\overline{\mathbf{B}\mathbf{f}'}(x_1)+\mathbf{d}g'(x_1)+\overline{\mathbf{d}g'}(x_1)\right]^-=-2\mathbf{t}_0(x_1)\end{aligned} \tag{36}$$

for $-a\leqslant x_1\leqslant a$. By the Liouville theorem (Rudin, 1987) we obtain that

$$\mathbf{B}\mathbf{f}'(z)-\overline{\mathbf{B}\mathbf{f}'}(z)+\mathbf{d}g'(z)-\overline{\mathbf{d}g'}(z)=0,\quad \text{for all } z \tag{37}$$

Then, the following conditions hold

$$\begin{aligned}&\mathbf{B}\mathbf{f}'(x_1)^+ + \mathbf{d}g'(x_1)^+ = \overline{\mathbf{B}\mathbf{f}'}(x_1)^+ + \overline{\mathbf{d}g'}(x_1)^+\\&\mathbf{B}\mathbf{f}'(x_1)^- + \mathbf{d}g'(x_1)^- = \overline{\mathbf{B}\mathbf{f}'}(x_1)^- + \overline{\mathbf{d}g'}(x_1)^-\end{aligned} \tag{38}$$

By using of Eq. (38), Eq. $(36)_2$ leads to

$$\mathbf{B}\left[\mathbf{f}'(x_1)^+ + \mathbf{f}'(x_1)^-\right]+\mathbf{d}\left[g'(x_1)^+ + g'(x_1)^-\right]=-\mathbf{t}_0(x_1),\quad -a\leqslant x_1\leqslant a \tag{39}$$

Once the function $g'(z)$ is known, Eq. (39) can be solved since $\mathbf{B}$ is non-singular. A solution which vanishes at infinity can be obtained

$$\mathbf{f}'(z)=\frac{\mathbf{X}(z)}{2\pi i}\int_{-a}^{a}\frac{\mathbf{X}^{-1}(x_1)^+}{x_1-z}\mathbf{p}(x_1)dx_1 \tag{40}$$

where

$$X(z_\alpha) = \left\langle \frac{1}{\sqrt{z_\alpha^2 - a^2}} \right\rangle, \quad \mathbf{p}(x_1) = -\mathbf{B}^{-1}[\mathbf{t}_0(x_1) + \mathbf{d}(g'(x_1)^+ + g'(x_1)^-)], \quad -a \leqslant x_1 \leqslant a \tag{41}$$

Similarly, by substituting Eqs. (19), (25), and (33) into Eqs. (40) and (41), the interaction SED function vectors can be obtained as (Li and Kardomateas, 2005)

$$\begin{aligned}\mathbf{f}'_{\text{int}}(z_\alpha) = &-\frac{T_0}{8\pi i}\left[\mathbf{y}(z_\alpha, z_{0t})\mathbf{B}^{-1}\mathbf{d} - \mathbf{y}(z_\alpha, \bar{z}_{0t})\mathbf{B}^{-1}\bar{\mathbf{d}}\right] \\ &+ \left[\sum_{k=1}^{4}\mathbf{Y}(z_\alpha, z_{0k})\mathbf{B}\mathbf{I}_k\mathbf{B}^{\mathrm{T}} - \sum_{k=1}^{4}\mathbf{Y}(z_\alpha, \bar{z}_{0k})\mathbf{B}\mathbf{I}_k\mathbf{B}^{\mathrm{T}}\right]\mathbf{b} \\ &+ \frac{T_0}{8\pi i}[\mathbf{F}(z_\alpha, z_{0t}) + \mathbf{F}(z_\alpha, \bar{z}_{0t})]\mathbf{B}^{-1}\mathbf{d}\end{aligned} \tag{42}$$

where

$$\begin{aligned}\mathbf{y}(z_\alpha, z_{0t}) &= \left\langle \left[1 - \frac{z_\alpha + \sqrt{z_{0t}^2 - a^2}}{\sqrt{z_\alpha^2 - a^2}}\right] \ln(z_\alpha - z_{0t}) \right\rangle \\ \mathbf{Y}(z_\alpha, z_{0k}) &= \frac{1}{4\pi i}\left\langle \frac{1}{\sqrt{z_\alpha^2 - a^2}} + \frac{1}{z_\alpha - z_{0k}}\sqrt{\frac{z_{0k}^2 - a^2}{z_\alpha^2 - a^2}} - \frac{1}{z_\alpha - z_{0k}} \right\rangle \mathbf{B}^{-1} \\ \mathbf{F}(z_\alpha, z_{0t}) &= \mathbf{y}(z_\alpha, z_{0t}) - \left\langle \left[1 - \frac{z_\alpha}{\sqrt{z_\alpha^2 - a^2}}\right] \ln\sqrt{z_{0t}^2 - a^2} \right\rangle\end{aligned} \tag{43}$$

4. Thermoelectroelastic fields induced by remote uniform loads

Consider a crack of length $2a$ in an infinite piezoelectric medium subjected to remote uniform heat flux load and uniform electro-mechanical loads

$$h_2 = h_2^\infty, \quad \mathbf{t}_2^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^{\mathrm{T}} \tag{44}$$

It is convenient to represent the solution due to h_2^∞ and $\mathbf{t}_2^\infty$ as the sum of a uniform SED fields in an unflawed medium and a corrective solution in which the main crack boundary conditions can be expressed as

$$\begin{aligned} h_2(x_1, x_2 = 0^+) &= h_2(x_1, x_2 = 0^-) = -h_2^\infty \\ \mathbf{t}_2(x_1, x_2 = 0^+) &= \mathbf{t}_2(x_1, x_2 = 0^-) = -\mathbf{t}_2^\infty \end{aligned} \tag{45}$$

Substitution of Eq. $(45)_1$ into (31), the corresponding heat potential for the above problem are given by

$$g''(z_t) = \frac{h_2^\infty}{2ik}\left[1 - \frac{z_t}{\sqrt{z_t^2 - a^2}}\right] \tag{46}$$

Integrating this equation leads to

$$g'(z_t) = \frac{h_2^\infty}{2ik}\left[z_t - \sqrt{z_t^2 - a^2}\right] \tag{47}$$

Substitution of Eq. (45) into (40) and (41), the SED potential can be obtained in the closed form

$$\begin{aligned}\mathbf{f}'(z_\alpha) = &-\frac{1}{2}\left\langle 1 - \frac{z_\alpha}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{B}^{-1}\mathbf{t}_2^\infty \\ &- \frac{h_2^\infty}{2ik}\left\langle z_\alpha - \frac{z_\alpha^2 - a^2/2}{\sqrt{z_\alpha^2 - a^2}} \right\rangle \mathbf{B}^{-1}\mathbf{d}\end{aligned} \tag{48}$$

5. Crack branching in anisotropic thermopiezoelectric medium

5.1. Singular integral equations

The branched portion of the crack can be modeled by continuous distribution of thermopiezoelectric dislocations with density $T_0(\xi)$ and $\mathbf{b}(\xi)$ along the line, $z = a + \eta\, z^*$, $z_0 = a + \xi\, z^*$, where $z^* = \cos\theta + p\sin\theta$ emanating from the main crack tip (Qin and Mai, 2000). Enforcing the satisfaction of heat flux and traction-charge free conditions on the branch, a system of singular integral equations for the dislocation density $T_0(\xi)$ and $\mathbf{b}(\xi)$ can be derived as

$$\frac{k}{2\pi}\int_0^c \frac{T_0(\xi)}{\eta - \xi}d\xi + \frac{k}{4\pi}\int_0^c K_t(\eta, \xi)T_0(\xi)d\xi + Q_t^\infty(\eta) = 0 \tag{49}$$

where

$$\begin{aligned} K_t(\eta, \xi) &= \mathrm{Re}\left\{ z_t^* \left[\frac{1}{\sqrt{z_t^2 - a^2}} + \frac{\sqrt{z_{0t}^2 - a^2}}{2\sqrt{z_t^2 - a^2}(z_t - z_{0t})} - \frac{1}{2(z_t - z_{0t})} + \frac{\sqrt{\bar{z}_{0t}^2 - a^2}}{2\sqrt{z_t^2 - a^2}(z_t - \bar{z}_{0t})} - \frac{1}{2(z_t - \bar{z}_{0t})} \right] \right\} \\ Q_t^\infty(\eta, \xi) &= h_2^\infty \mathrm{Re}\left\{ z_t^* \left[1 - \frac{z_t}{\sqrt{z_t^2 - a^2}} \right] \right\} \\ z_t^* &= \cos\theta + p_*\sin\theta \end{aligned} \tag{50}$$

and

$$\frac{\mathbf{L}}{2\pi}\int_0^c \frac{\mathbf{b}(\xi)}{\eta - \xi}d\xi + \int_0^c \mathbf{K}_b(\eta, \xi)\mathbf{b}(\xi)d\xi + \int_0^c \mathbf{K}_{bt}(\eta, \xi)T_0(\xi)d\xi + \mathbf{Q}_{bt}^\infty(\eta) + \mathbf{Q}_b^\infty(\eta) = 0 \tag{51}$$

where

$$\begin{aligned} \mathbf{L} &= -2i\mathbf{B}\mathbf{B}^T \\ \mathbf{K}_b(\eta, \xi) &= 2\mathrm{Re}\left\{ \mathbf{B}\langle z_\alpha^* \rangle \left[\sum_{k=1}^{4}\mathbf{Y}(z_\alpha, z_{0k})\mathbf{B}\mathbf{I}_k\mathbf{B}^{\mathrm{T}} - \sum_{k=1}^{4}\mathbf{Y}(z_\alpha, \bar{z}_{0k})\mathbf{B}\mathbf{I}_k\mathbf{B}^{\mathrm{T}} \right] \right\} \\ \mathbf{K}_{bt}(\eta, \xi) &= \mathrm{Re}\left\{ -\frac{1}{4\pi i}\mathbf{B}\langle z_\alpha^* \rangle \left[\mathbf{y}(z_\alpha, z_{0t})\mathbf{B}^{-1}\mathbf{d} - \mathbf{y}(z_\alpha, \bar{z}_{0t})\mathbf{B}^{-1}\bar{\mathbf{d}} \right] \right\} \\ &\quad + \mathrm{Re}\left\{ \frac{1}{4\pi i}\mathbf{B}\langle z_\alpha^* \rangle [\mathbf{F}(z_\alpha, z_{0t}) + \mathbf{F}(z_\alpha, \bar{z}_{0t})]\mathbf{B}^{-1}\mathbf{d} \right\} \\ &\quad + \mathrm{Re}\left[\frac{1}{2\pi i}\mathbf{d} z_t^* B(z_t, z_{0t}) \right] + \frac{1}{\pi}\mathrm{Im}\left[\mathbf{d}\, z_t^* \ln(z_t - z_{0t}) \right] \\ \mathbf{Q}_{bt}^\infty(\eta, \xi) &= \frac{h_2^\infty}{k}\mathrm{Im}\left[\mathbf{d}\, z_t^*(z_t - \sqrt{z_t^2 - a^2}) \right] \\ \mathbf{Q}_b^\infty(\eta, \xi) &= \mathrm{Re}\left\{ -\mathbf{B}\left\langle z_\alpha^* \left[1 - \frac{z_\alpha}{\sqrt{z_\alpha^2 - a^2}} \right] \right\rangle \mathbf{B}^{-1}\mathbf{t}_2^\infty - \frac{h_2^\infty}{ik}\mathbf{B}\left\langle z_\alpha^* \left[z_\alpha - \frac{z_\alpha^2 - a^2/2}{\sqrt{z_\alpha^2 - a^2}} \right] \right\rangle \mathbf{B}^{-1}\mathbf{d} \right\} \\ z_\alpha^* &= \cos\theta + p_\alpha\sin\theta \end{aligned} \tag{52}$$

where "Re" and "Im" strand for the real part and imaginary part of a complex number, respectively.

For the purpose of numerical calculation, the following normalized quantities are introduced

$$x = \frac{2\eta - c}{c}, \quad t = \frac{2\xi - c}{c} \tag{53}$$

where $|x| < 1$ and $|t| < 1$, then Eqs. (49) and (51) can be rewritten as

$$\frac{k}{2\pi}\int_{-1}^{1} \frac{T_0(t)}{x - t}dt + \frac{kc}{8\pi}\int_{-1}^{1} \tilde{K}_t(x, t)T_0(t)dt + \tilde{Q}_t^\infty(x) = 0 \tag{54}$$

$$\frac{\mathbf{L}}{2\pi}\int_{-1}^{1}\frac{\mathbf{b}(t)}{x-t}dt+\frac{c}{2}\int_{-1}^{1}\tilde{\mathbf{K}}_b(x,t)\mathbf{b}(t)dt+\frac{c}{2}\int_{-1}^{1}\tilde{\mathbf{K}}_{bt}(x,t)T_0(t)dt + \tilde{\mathbf{Q}}_{bt}^{\infty}(x)+\tilde{\mathbf{Q}}_b^{\infty}(x)=0 \quad (55)$$

where $\tilde{K}_t(x,t)$, $\tilde{\mathbf{K}}_b(x,t)$, $\tilde{\mathbf{K}}_{bt}(x,t)$, $\tilde{\mathbf{Q}}_t^{\infty}(x)$, $\tilde{\mathbf{Q}}_{bt}^{\infty}(x)$ and $\tilde{\mathbf{Q}}_b^{\infty}(x)$ are obtained by substituting Eq. (53) into (50) and (52), respectively.

5.2. Numerical scheme

This system of singular equations involves two unknowns, $T_0(t)$ and $\mathbf{b}(t)$ which coupled through the term $\tilde{\mathbf{K}}_{bt}(x,t)$ in Eq. (55). The singular integral Eqs. (54) and (55) can be solved numerically using a method developed by Erdogan and Gupta (1972). Accordingly, the unknown density functions, $T_0(t)$ and $\mathbf{b}(t)$ in the Eqs. (54) and (55) can be written in the form

$$T_0(t)=\frac{\tilde{T}_0(t)}{\sqrt{1-s^2}}=\frac{\sum_{j=0}^{n-1}\tilde{T}_{0j}T_j(t)}{\sqrt{1-s^2}},\quad \mathbf{b}(t)=\frac{\tilde{\mathbf{b}}(t)}{\sqrt{1-s^2}}=\frac{\sum_{j=0}^{n-1}\tilde{\mathbf{b}}_jT_j(t)}{\sqrt{1-s^2}} \quad (56)$$

where $\tilde{T}_0(t)$ and $\tilde{\mathbf{b}}(t)$ are regular functions defined in the interval $|t|<1$, $T_j(t)$ is the Chebyshev polynomial of the first kind, and the coefficients $\tilde{T}_{0j}$ and $\tilde{\mathbf{b}}_j$ are constants as yet to be determined. It is noted that the singularity at the branch root is not $-1/2$ and considered to be no larger than $1/2$ based on the singularity analysis (Bogy, 1971; Keer and Miller, 1982; Li and Kardomateas, 2005). But in order to facilitate the solution procedure, the square-root singularity is assumed in this paper, and (Qin and Mai, 2000)

$$\tilde{T}_0(-1)=0 \quad \text{and} \quad \tilde{\mathbf{b}}(-1)=0 \quad (57)$$

Therefore, the Eqs. (54) and (55) can be rewritten in numerical form as

$$\sum_{m=1}^{n}\frac{k}{n}\left[\frac{1}{2(x_r-t_m)}-\frac{c}{8}\tilde{K}_t(x_r,t_m)\right]\tilde{T}_0(t_m)+\tilde{Q}_t^{\infty}(x_r)=0 \quad (58)$$

$$\sum_{m=1}^{n}\frac{1}{2n}\left[\frac{\mathbf{L}}{x_r-t_m}+\pi c\tilde{\mathbf{K}}_b(x_r,t_m)\right]\tilde{\mathbf{b}}(t_m)+\frac{c}{2}\int_{-1}^{1}\tilde{\mathbf{K}}_{bt}(x_r,t)T_0(t)dt + \tilde{\mathbf{Q}}_{bt}^{\infty}(x_r)+\tilde{\mathbf{Q}}_b^{\infty}(x_r)=0 \quad (59)$$

where

$$t_m=\cos\left[\frac{(2m-1)\pi}{2n}\right],\quad (m=1,2,n)$$
$$x_r=\cos\left[\frac{r\pi}{n}\right],\quad (r=1,2,n-1) \quad (60)$$

From Eqs. $(57)_1$ and (58), $T_0(t)$ can be solved uniquely, once the solution for T_0 is obtained, one can move to solve Eq. (59). The solution of integral equation (51) has the square-root singularity at both crack tips. Therefore, the numbers of integration points and collocation points take n and $n-1$ as shown from Eq. (60), respectively. The Eqs. (59) and $(57)_2$ provided a system of $4n$ linear algebraic equations, and the dislocation density $\mathbf{b}$ can be solved uniquely. Once the function $\tilde{\mathbf{b}}(t)$ has been found, the SED, $\mathbf{\Pi}_2(x)$, in a coordinate system local to the crack branch line can be expressed as in the form

$$\mathbf{\Pi}_2(x)=\mathbf{\Omega}(\theta)\left[\frac{\mathbf{L}}{2\pi}\int_{-1}^{1}\frac{\mathbf{b}(t)}{x-t}dt+\frac{c}{2}\int_{-1}^{1}\tilde{\mathbf{K}}_b(x,t)\mathbf{b}(t)dt + \frac{c}{4\pi}\int_{-1}^{1}\tilde{\mathbf{K}}_{bt}(x,t)T_0(t)dt+\tilde{\mathbf{Q}}_{bt}^{\infty}(x)+\tilde{Q}_b^{\infty}(x)\right] \quad (61)$$

where the 4×4 matrix $\Omega(\theta)$ whose components are the cosine of the angle between the local coordinates and the global coordinates is

$$\mathbf{\Omega}(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0 & 0\\ -\sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \quad (62)$$

5.3. The SED intensity factors and energy release rate

The SED intensity factors at the right tip of the branch crack are of interest and can be numerically calculated as

$$\mathbf{K}=[K_{\mathrm{II}},\ K_{\mathrm{I}},\ K_{\mathrm{III}},\ K_{\mathrm{D}}]^{\mathrm{T}}=\lim_{x\to1^+}\sqrt{\pi c(x-1)}\mathbf{\Pi}_2(x) = \sqrt{\frac{\pi c}{8}}\mathbf{\Omega}(\theta)\mathbf{L}\tilde{\mathbf{b}}(1) \quad (63)$$

where the following identity

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{x-t}\frac{T_i(t)}{\sqrt{1-t^2}}ds=\frac{|x|}{x\sqrt{x^2-1}}\left[x-\frac{|x|}{x}\sqrt{x^2-1}\right]^i,\quad |x|>1,\ i=0,1,2,\ldots \quad (64)$$

is used. Moreover, the energy release rate of the crack branch can be obtained by using the following expression (Park and Sun, 1995b)

$$G=\frac{1}{2}\mathbf{K}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{K} \quad (65)$$

6. Numerical results and discussion

The influence of thermal conductivity and electric field on the crack branching in the thermopiezoelectric (PZT-5H) medium is investigated in this section. The material properties for PZT-5H are as follows (Tsamasphyros and Song, 2005):

Elastic constants:

$$c_{11}=126\ \mathrm{GPa},\quad c_{12}=79.5\ \mathrm{GPa},\quad c_{13}=84.1\ \mathrm{GPa},\quad c_{33}=117\ \mathrm{GPa},\quad c_{44}=23\ \mathrm{GPa}$$

Piezoelectric constants:

$$e_{31}=-6.5\ \mathrm{C/m^2},\quad e_{33}=23.3\ \mathrm{C/m^2},\quad e_{15}=17\ \mathrm{C/m^2}$$

Dielectric constants:

$$\kappa_{11}=151\times10^{-10}\ \mathrm{F/m},\quad \kappa_{33}=130\times10^{-10}\ \mathrm{F/m}$$

Heat conduction constants:

$$k_{11}=50\ \mathrm{W/km},\quad k_{33}=75\ \mathrm{W/km},\quad k_{13}=0$$

Thermal-stress coefficient:

$$\beta_{11}=1.9738\times10^6\ \mathrm{N/km^2},\quad \beta_{33}=1.4165\times10^6\ \mathrm{N/km^2}$$

Pyroelectric constants:

$$\gamma_3=-5.4831\ \mathrm{C/km^2}$$

The pure tension $\sigma_{22}^{\infty}=1$ Mpa is the specified applied load in the following numerical calculations, and a coefficient α is defined as

$$\alpha=\frac{h_2^{\infty}a\beta_{33}}{k_{33}\sigma_{22}^{\infty}} \quad (66)$$

The results for branched crack tip SED intensity factors with the different branch lengths are plotted in Figs. 2–4. The angles of the branch are chosen to be 0, $\pi/6$ and $\pi/4$, the electric displacement $D_2^{\infty}=0.001\ \mathrm{C/m^2}$, $\alpha=1$ and $K_0=\sigma_{22}^{\infty}\sqrt{\pi a}$. From the Figs. 2–4, it is shown that the SED intensity factors at the tip of a vanishingly small branch become independent of the small length c. The same conclusion was obtained by Zhu and Yang (1999) for the problem

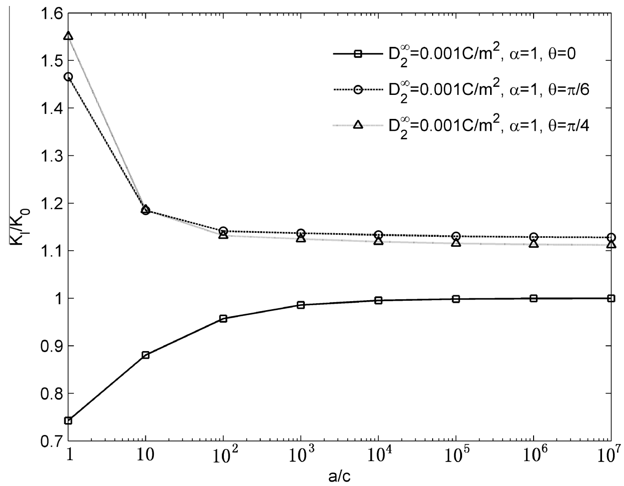

Fig. 2. Variation of branch tip K_I with branch length a/c.

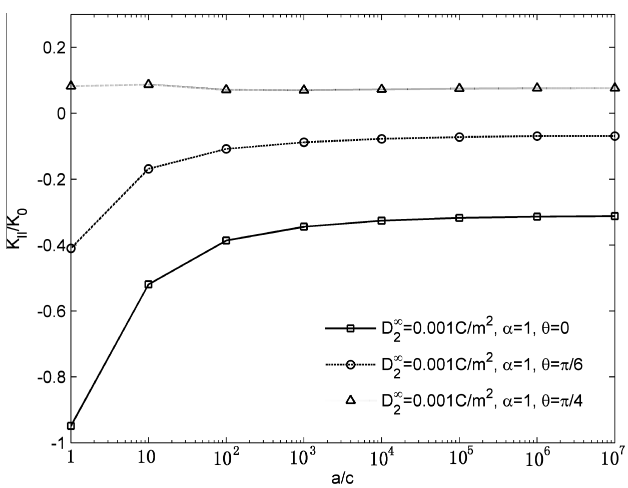

Fig. 3. Variation of branch tip K_{II} with branch length a/c.

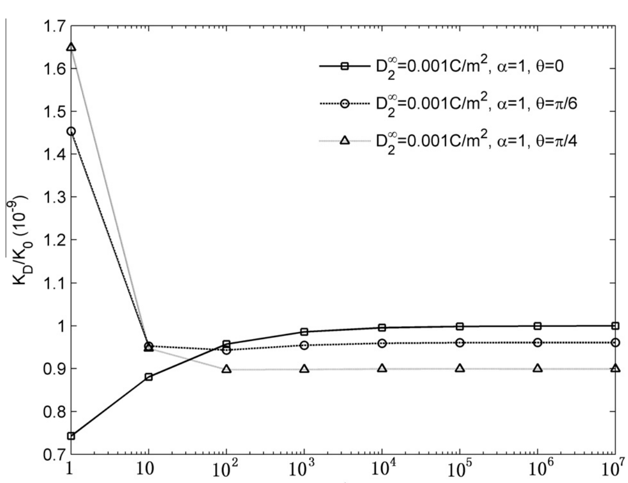

Fig. 4. Variation of branch tip K_D with branch length a/c.

of crack kinking in a piezoelectric medium. As expected, the electric field and heat flux have no influence on the mode-I stress intensity factor when $\theta = 0$ due to symmetry of heat load. For $\theta \neq 0$ (e.g. in Figs. 2–4, $\theta = \pi/6$ and $\pi/4$), the effect on K_I made by the far field uniform heat flux is great. Here, the logarithmic coordinates is used for x_1-axis, i.e., the value of a/c is varying from 1 to 10^7. In the following calculations, a piezoelectric crack with an "infinitesimal" branch length, $a/c = 10^6$ is assumed. The variation of branch tip K_I, K_{II} and K_D with branch angle θ are shown from Figs. 5–7. It can be found that K_{II} always increases with increasing value of θ, D_2^∞ has a little effect on the mode II stress intensity factor K_{II} from Fig. 6. Fig. 7 shows that the value of K_D under heat flux loading is bigger than it without heat flux loading due to the pyroelectric effect.

Different criteria have been proposed to predict the direction of crack branching. The branching angle at which the K_I attains its maximum value (K_{II} reaching its minimum value) coincide with the angle which makes the energy release rate attain its maximum value in an anisotropic material (Li and Kardomateas, 2005). For piezoelectric materials, it is shown that the intensity factors based criterion, in some cases, does not yield the same result as the one obtained by an energy based criterion (see Pak, 1992; Zhang et al., 1998). For the bimaterial systems, the crack propagation criterion such as comparing the ratio of the energy release rate for penetrating the interface and for deflecting into the interface to the ratio of the mode I toughness of material to the interface toughness was always used (Qin and Zhang, 2000). So the energy release rate

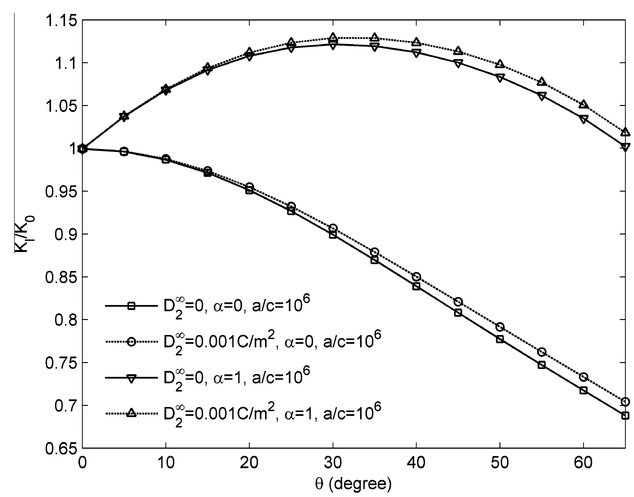

Fig. 5. Variation of branch tip K_I with branch angle θ.

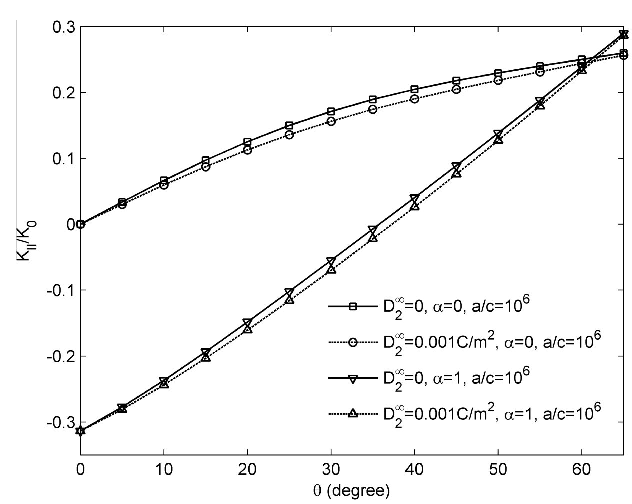

Fig. 6. Variation of branch tip K_{II} with branch angle θ.

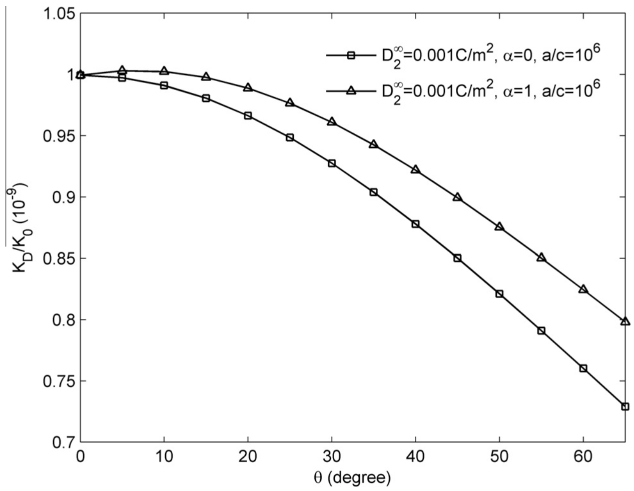

Fig. 7. Variation of branch tip K_D with branch angle θ.

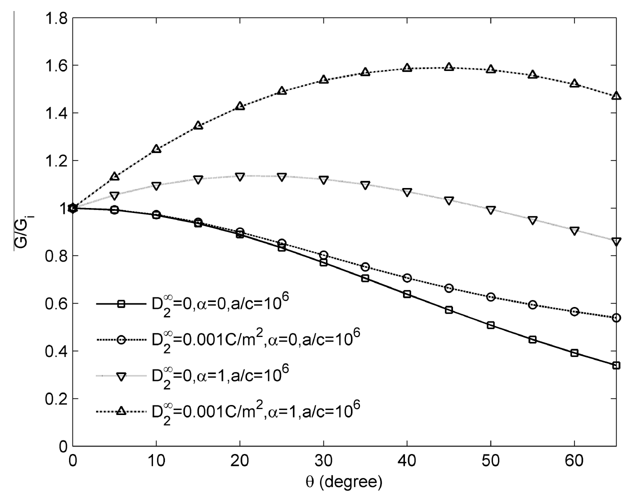

Fig. 8. Variation of energy release rate G/G_i with branch angle θ.

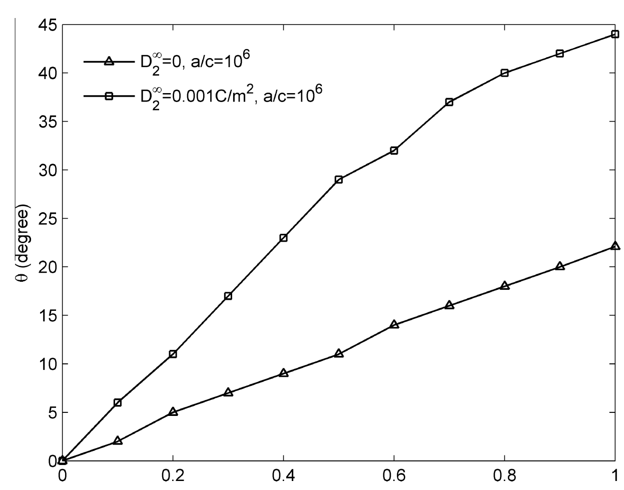

Fig. 9. Crack branch direction θ versus α.

criteria should be used for thermopiezoelectric crack branching problem. It can be seen that the crack will propagate in a straight line under a tensile stress and a positive electric field from the Figs. 5 and 8, where G_i is the energy release rate without branching of the corresponding case. Fig. 8 also shows that the branch angle is affected by the electric displacement load D_2^∞ when heat flux is applied. Generally, the critical branch angle will increase along with the increase of value of D_2^∞. In particular, when $D_2^\infty = 0$, $\alpha = 1$, the $G_{\max}/G_i$ attains its maximum value with a corresponding branching angle $\theta = 22.1^\circ$; while $D_2^\infty = 0.001\ \mathrm{C/m^2}$, $\alpha = 1$, the $G_{\max}/G_i$ attains its maximum value with a corresponding branching angle $\theta = 43.9^\circ$ for the PZT-5H. The assessment of crack growth direction is very important for the crack branching problem (Qin and Mai, 1997). Fig. 9 shows the numerical results of θ versus the thermal-stress coefficient α under the different electric displacement D_2^∞. It can be seen that crack branching direction θ varies approximately linearly with the α for $D_2^\infty = 0$. For the case of $D_2^\infty = 0.001\ \mathrm{C/m^2}$, θ increases monotonically, but the trend slows down with the increasing value of α .

7. Conclusion

Thermopiezoelectric crack branching of piezoelectric materials is studied based on extended Stroh formalism and continuous distribution of dislocation approach. In this investigation, an explicit Green's function to the interaction between a thermal-piezoelectric dislocation and a crack is obtained. Two set of coupled singular integral equations are derived for the thermal dislocation and piezoelectric dislocation density functions associated with a branched crack. As a result, the formulation for the stress and electric displacement intensity factors and energy release rate can be expressed in terms of the dislocation density function and the branch angle. Numerical results indicate that the crack tends to propagate in a straight line under a tensile stress and an electric field. The analysis also shows that the critical branch angle increases with increasing value of electric displacement when heat flux load is applied.

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